RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 55, NUMBER 5 MAY 1997

Centrifugal space-charge force of an electron beam in a focusing element

Bruce E. Carlsten
Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 27 January 1997)

The centrifugal space-charge force of a continuous electron beam, focused in a lens of an accelerator, is numerically calculated. As in the case of an electron beam in a dipole magnetic field, the effect on the transverse motion of the particles from the centrifugal space-charge force in a focusing element tends to cancel the effect from the potential depression of the beam; however, the cancellation is not exact, as it is for the dipole case. The centrifugal space-charge force in the focusing case arises from a nonzero axial derivative of the transverse vector potential due to the beam's space charge, as the beam is transversely accelerated in the focusing element. The transverse equation of motion for particles in the beam is used to quantify the partial cancellation of the nonlinear transverse acceleration from the centrifugal space-charge force and from the beam's potential depression, for focusing both with a magnetic quadrupole lens and a solenoid.
[S1063-651X(97)51505-4]

PACS number(s): 29.27.Bd, 03.50.De, 41.75.Ht

In 1986, Talman [1] described a previously unconsidered space-charge force that exists if a continuous beam is deflected in a uniform dipole magnetic field. This space-charge force, called the centrifugal space-charge force (CSCF), does not exhibit the usual relativistic cancellation, and there was concern that it would lead to a substantial amount of beam quality degradation, particularly for circular machines at high energy. However, a subsequent investigation of this effect led Lee in 1990 to observe that the effect from the centrifugal space-charge force was primarily to cancel the effect from the potential depression of the beam [2]; and, in fact, less beam quality degradation results than if this force did not exist. Careful expansions of the transverse equation of motion demonstrating this cancellation can be found in Refs. [2] and [3]. The cancellation arises because the radial derivative of the vector potential in the direction of motion due to the beam's space charge leads to a term that depends on the deviation of a particle's potential from the potential at the center of the beam.

Some amount of cancellation between the CSCF and the beam's potential depression is not surprising. Naively, one would expect that a particle's bending radius in a uniform dipole field depends only on that particle's kinetic energy; however, there is a certain amount of momentum stored in the Coulomb space-charge fields surrounding a bunch, which also must be bent through some interaction with the particles themselves. The exact first order (in terms of the beam radius divided by the radius of curvature) cancellation of the transverse forces for all particles is surprising, and tells us that the amount of inertia in the Coulomb fields that each particle must overcome is equal to the potential depression of that particle divided by the speed of light squared (to first order). As a result, all particles in a beam are deflected by a dipole magnetic field as if they have the potential associated with the beam pipe wall, and not the depressed potential associated with their kinetic energy. This effect was recently verified experimentally by comparing the deflection of a 6-MeV, 300-A electron beam with that of a 4-kA electron beam [4].

When a very high-brightness electron beam is within a focusing element along an accelerator, the beam quality can potentially degrade, resulting from the energy variation of particles which arises from the potential depression of the beam. For example, within an ideal quadrupole (which has linear transverse forces), particles with slightly lower energy (those at the center of the beam) ''see'' a shorter focal length than particles with slightly higher energy (those at the radial edge of the beam), resulting in an axially smeared out focus, and an emittance growth of the beam. There is, however, a focusing analog of the CSCF, which arises from a nonzero axial derivative of the transverse vector potential as the beam is transversely accelerated. The resulting azimuthal magnetic field tends to reduce the focal length variation and resulting beam quality degradation. Because the transverse acceleration in a focusing element is very nearly proportional to transverse position, the cancellation between the potential depression and the focusing CSCF is not exact to first order.

For a solenoid, the focusing CSCF will actually lead to an increase in the beam's emittance, instead of decreasing the emittance growth as in a quadrupole. This is because of an additional effect from the beam's self-diamagnetic axial field, which itself counters the effect of the beam's potential depression. Within a solenoid, the electron beam rotates azimuthally, leading to a counteraxial magnetic field in the center of the beam. This reduced axial field leads to less azimuthal rotation of the beam (the reduction is more pronounced at the center of the beam, and there is no reduction at the radial edge of the beam), and less radial focusing force. For a uniform density electron beam, the decrease in focusing force is exactly the same as the decrease in the beam's potential energy, and, in the absence of the focusing CSCF, the focal length from a solenoid is the same at all radii within the beam. Now the effect of the focusing CSCF is to create a variation in the focal length across the beam, with a resulting emittance growth. We will now quantify (1) the focusing CSCF and (2) the nonlinearity introduced in the radial equation of motion by it, for solenoidal focusing of a uniform density beam.

The radial equation of motion of a particle within the central part of an ideal solenoid (where the magnetic field from the solenoid is purely axial and uniform, and where we are additionally assuming that the geometry is axisymmetric) is given by

$$m\frac{d(\gamma\dot{r})}{dt}=eE_r+e(\nu_\theta B_{\rm dia}-\nu_z B_\theta)+e\nu_\theta B_{\rm ext}+\frac{\gamma m\nu_\theta^2}{r}, \tag{1}$$

where γ is the relativistic mass factor, $B_{\rm ext}$ is the axial magnetic field from the solenoid (including the diamagnetic effect from the image currents in the beampipe), $B_{\rm dia}$ is the induced diamagnetic axial magnetic field from the beam current opposing the solenoidal field, B_θ is the azimuthal magnetic field from the space charge, and E_r is the radial electric field from the space charge, all at the position of the particle, and e and m are the electronic charge and mass, respectively. Most of these terms are mostly linear with radius—the parts of the terms that are not lead to the variation in focal lengths. The $\nu_z B_\theta$ term mostly cancels E_r (to order $1/\gamma^2$), but also includes the focusing CSCF. There is also a potential depression within the beam (a variation of γ that is a function of the radius). Our approach will be to expand the radial equation of motion in terms of the variation of γ, to lowest order, in order to quantify the focal length variation.

The azimuthal velocity is found by application of Busch's theorem [5] (the conservation of angular momentum):

$$\nu_\theta=-\frac{e}{\gamma m r}\int_0^r(B_{\rm ext}+B_{\rm dia})r\,dr. \tag{2}$$

The relativistic mass factor is given by $\gamma(r)=\gamma_a+\gamma_1(r)$, where γ_a is the mass factor along the axis ($r=0$). For a beam of uniform density ρ_0,

$$\gamma_1=\frac{e}{4mc^2}\frac{\rho_0}{\varepsilon}r^2,$$
$$\frac{B_{\rm dia}}{B}=-\left(\frac{\gamma_b}{\gamma_a}-\frac{\gamma_1}{\gamma_a}\right), \tag{3}$$

where we have now introduced γ_b as the difference in the relativistic mass factor between the center and the radial edge of the beam, and B as the total axial magnetic field. This leads to

$$B=B_a\left(1+\frac{\gamma_1}{\gamma_a}\right),\quad \gamma=\gamma_a\left(1+\frac{\gamma_1}{\gamma_a}\right), \tag{4}$$

where B_a is the axial field at the axis of symmetry. Note that γ_1 depends quadratically on the beam radius, and is positive.

The beam-induced azimuthal magnetic field in Eq. (1) is given in terms of the vector potential by

$$B_\theta=\frac{\partial}{\partial z}A_r-\frac{1}{r}\frac{\partial}{\partial r}rA_z. \tag{5}$$

Note that

$$\phi=\int\frac{\rho}{4\pi\varepsilon r}d\vec{r},\quad \vec{A}=\int\mu\frac{\vec{j}}{4\pi r}d\vec{r}, \tag{6}$$

where ϕ is the scalar potential, $\vec{j}$ is the current density, and the integrals are over all space. We will assume that the beam flow is laminar, and the current density is

$$\vec{j}=\rho\beta c(\hat{z}\cos\Psi+\hat{r}\sin\Psi), \tag{7}$$

where the angle Ψ is given by

$$\Psi=\frac{r}{r_b}\frac{dr_b}{dz}, \tag{8}$$

r_b is the beam edge radius, and dr_b/dz is the angle of divergence of the beam edge. Note that the difference between the axial vector potential and $(\beta/c)\phi$ is on the order of Ψ^2, but the radial vector potential scales as $(\beta/c)\phi)$ times Ψ. At this point, we will assume that the angle of the beam convergence is much smaller than $1/\gamma$ (if this is not true, another, although smaller term, needs to be kept in the following derivation). This means that the space-charge electric field and the space-charge magnetic field from the axial vector potential still cancel to order $1/\gamma^2$ to first order in Ψ, and the additional space-charge force introduced by the radial vector potential adds in a force of order Ψ that is not canceled by other fields.

The vector potential for a uniform density beam at a location y along the vertical axis is given by

$$\vec{A}=\int_{-z}^{z}d\zeta\int_0^{2\pi}d\theta\int_0^{r_b}\mu\times\frac{I(\sin\theta\,\sin\Psi,0,\cos\Psi)}{4\pi^2r_b^2\sqrt{\zeta^2+r^2+y^2-2ry\sin\theta}}r\,d\vec{r} \tag{9}$$

where additionally the beam radius is given as a function of axial position from the observer location ζ by

$$r_b=r_o+\zeta\tan\Psi, \tag{10}$$

and where r_o is the beam radius at the observer location.

As is well known [6], the integral for the axial vector potential diverges as the limit of integration z approaches infinity—this, however, is not the case for the radial vector potential, which quickly reaches a maximum. This is easily understood by observing that for large axial displacements, the radical part of the denominator in Eq. (9) becomes $\sqrt{\zeta^2}$, and the $\sin\theta$ term in the numerator then integrates to zero (far from the observer location, the different radial velocities of the beam current density average to zero).

We can normalize the radial vector potential to the scalar potential at the beam radius $\phi_b=\gamma_b mc^2/e$,

$$A_r=\frac{dr}{dz}\chi\frac{\beta}{c}\phi_b=\frac{dr}{dz}\chi\mu\frac{I}{4\pi}, \tag{11}$$

where now χ is a parameter between zero and unity, depending on the beam geometry, divergence, and position within the beam, and I is current.

In Fig. 1 we plot $\chi(dr/dz)$ at the beam radius versus the axial limit of integration, normalized to the beam radius, for a divergence of 10 mrad. We see that the radial vector potential does indeed quickly reach its asymptotic value (for an axial limit of integration of about ten beam radii), and that $\chi\approx0.5$. In Fig. 2, we plot the asymptotic value of $\chi(dr/dz)$ as a function of radial position within the beam. In Fig. 3, we plot the asymptotic value of $\chi(dr/dz)$ at the beam

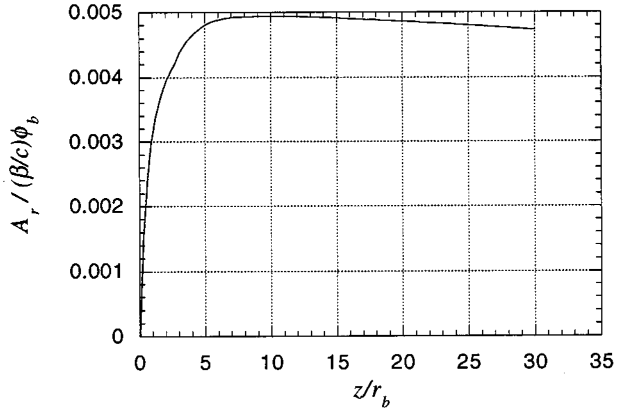

FIG. 1. Radial vector potential at the beam edge (normalized to the beam potential energy depression) vs the axial length of integration, for an edge divergence of 0.01 rad.

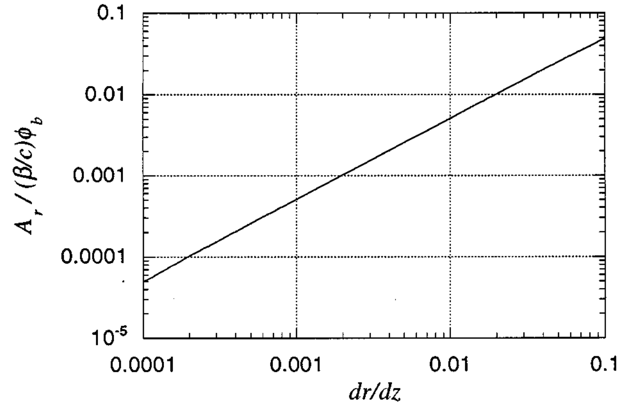

FIG. 3. Radial vector potential at the beam edge (normalized to the beam potential energy depression) vs edge divergence.

radius as a function of beam edge divergence, and see that $\chi\approx 0.5$ for a wide range of beam edge divergences. From Figs. 2 and 3, we see that $\chi=(13-8r/r_b)/10$ is a fair approximation within the beam ($r<r_b$).

For small rates of change of the beam edge diveregence, we find, from Eqs. (5) and (11),

$$B_\theta=-\frac{\beta}{c}\frac{1}{r}\frac{\partial}{\partial r}(r\phi)+\frac{\beta}{c}\left(\frac{13-8r/r_b}{10}\right)\frac{d^2r}{dz^2}\,\phi_b. \qquad (12)$$

The first term on the right-hand side leads to the $1/\gamma^2$ cancellation of the radial space-charge force. The second term is the focusing CSCF. Note that it vanishes if the beam convergence angle is a constant (if the beam is not being focused or defocused).

At this point, we have written out all the terms of the right-hand side of the radial equation of motion, and we can evaluate the nonlinear terms. For finding the radial dependence of the focusing focal length, we want the radial divergence instead of the radial velocity, so we still need to change the variable of differentiation on the left-hand side of the radial equation of motion. Using dots to refer to time derivatives and primes to refer to axial derivatives, we have

$$\frac{d}{dt}\,\gamma\dot r=\dot r\,\frac{d\gamma}{dt}+\gamma\ddot r=\frac{eE_r}{mc^2}\dot r^2+\gamma\ddot r \qquad (13)$$

and

$$\dot r=r'\,\nu_a\left(1+\frac{\nu(r)}{\nu_a}\right),$$
$$\ddot r=r''\,\nu_z^2(r)=r''\,\nu_a^2\left(1+2\,\frac{\nu(r)}{\nu_a}\right), \qquad (14)$$

where we define an average axial velocity ν_a and a relative axial velocity ν to be $\nu_z(r)=\nu_a+\nu(r)$. After calculating the focusing and the centrifugal acceleration terms, and combining the r'' terms and dividing through by a factor of γ, Eq. (1) becomes

$$r''m\,\nu_a^2\left(1+2\,\frac{\nu}{\nu_a}+\frac{\gamma_b}{\gamma_a}\left(\frac{13-8r/r_b}{10}\right)\right)$$
$$=\frac{eE_r}{\gamma^3}-\frac{e^2B_a^2r}{4\gamma^2m}\left(1+\frac{\gamma_1}{2\gamma_a}\right)\left(1+\frac{3\gamma_1}{2\gamma_a}\right)-\frac{e\,\nu_a^2E_r r'^2}{c^2\gamma}. \qquad (15)$$

Note that the radial dependence of the magnetic field terms and the mass factor will cancel in the second term on the right side of the equation (to first order in γ_1/γ_a), leading to a focusing term that is purely linear in radius. We will also assume at this point that the electric field terms can be ignored (either because of a small enough beam current or high enough energy and small beam convergence), and that the beam is at a high enough energy that the deviation in axial velocity can be ignored. Then, using the definition of the focal length of a solenoid for the beam near the axis,

$$f=\frac{r\,\gamma_a^2m^2\,\nu_a^2}{le^2B_a^2}, \qquad (16)$$

the radial divergence after a length l becomes

$$r'=-\frac{r}{f}\left(1-\frac{\gamma_b}{\gamma_a}\left(\frac{13-8r/r_b}{10}\right)\right)\quad(\text{solenoid}). \qquad (17)$$

The constant term within the nested parentheses makes the

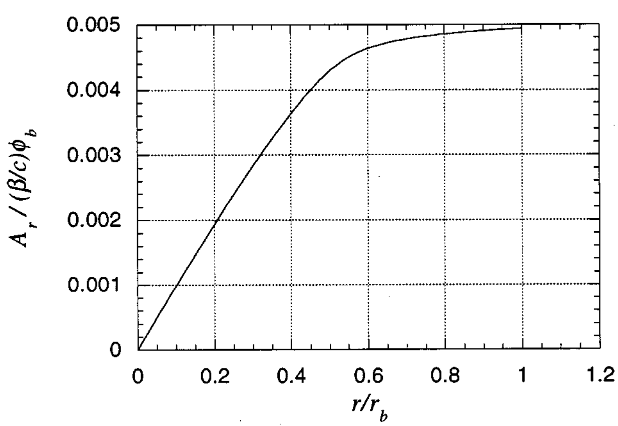

FIG. 2. Radial vector potential (normalized to the beam potential energy depression) vs radial position within the beam, for an edge divergence of 0.01 rad.

average focal length longer than one would expect from just considering the potential depression of the beam, in agreement with the concepts introduced from the CSCF in a dipole magnetic field. In addition, the term linear in r within the parentheses introduces a focal length variation transversely across the beam.

If a series of quadrupoles giving net focusing in both transverse planes is used, the effect from the focusing CSCF in a single plane adds and subtracts as the beam is focused or defocused. The net effect is then due to the net focusing, and the transverse equation of motion becomes (for equal net focal lengths in both transverse directions)

$$r' = -\frac{r}{f}\left(1 - \frac{\gamma_1(r)}{\gamma_a} - \frac{\gamma_b}{\gamma_a}\left(\frac{13-8r/r_b}{10}\right)\right) \quad \text{(quadrupole)}, \tag{18}$$

where now the focal length f is the net focal length at the center of the beam from the series of quadrupole lenses. As in the case for a solenoid, the constant term introduced by the focusing CSCF will increase the average focal length of the lens. The linear term will now tend to counter the variation in focal lengths introduced by the potential depression term, γ_1/γ_a $(=\gamma_b r^2/\gamma_a r_b^2$ for a round beam). Note that the variation in focal lengths is on the order of the potential depression divided by the beam potential for both a solenoid and a quadrupole lens. For a quadrupole lens, the variation in focal lengths is reduced by more than half by the focusing CSCF, demonstrating a similar cancellation as in the dipole magnet case.

This work was supported by funds from the Laboratory-Directed Research and Development program at Los Alamos National Laboratory, operated by the University of California for the U.S. Department of Energy.

[1] R. Talman, Phys. Rev. Lett. **56**, 1429 (1986).
[2] E. Lee, Part. Accel. **25,** 241 (1990).
[3] B. E Carlsten and T. O. Raubenheimer, Phys. Rev. E **51**, 1453 (1995).
[4] P. Allison, D. Moir, and G. Sullivan, Los Alamos National Laboratory report DARHT Technical Note No. 58, 1996 (unpublished); T. P. Hughes, Mission Research Company Report No. MRC/ABQ-N-576, 1996 (unpublished). In this experiment, steering coils were calibrated with low beam current. At high beam current, the beam centroid steered as if the effective average beam energy was at the wall potential instead of the depressed beam potential.
[5] J. F. Gittens, *Power Travelling-wave Tubes* (American Elsevier, New York, 1965).
[6] See, for example, P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970).